

# Satellite Libration on an Elliptic Orbit

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## Abstract -

The libration of a gravitationally oriented rigid satellite on an elliptic orbit is studied by the method of canonical transformation. The possibility of the center of mass of a satellite traveling on an elliptic orbit is examined. The transformations are performed in a mathematical way according to an operational method of perturbation conserving Hamiltonian formulism. The orbit eccentricity effect upon the amplitude and the frequency shift of the libration are exhibited. The ultimate nature of satellite libration on an elliptic orbit is revealed.

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Nomenclature -

$A, B, C$	Principal, central moments of inertia of the satellite.
$a$	Semi-major axis.
$e$	Eccentricity.
$H, H'$	Hamiltonian.
$h$	Integral constant.
$I$	Moment of inertia about $r$
$\mathcal{I}$	Inertia tensor.
$\underline{i}, \underline{j}, \underline{k}$	Unit vectors along principal axes.
$K$	Gravitational parameter of the center of force.
$\mathcal{L}$	Langrangian.
$L$	Action variable.
$l$	Angle variable.
$m$	Mass of satellite.
$\underline{n}$	Unit vector along
$p$	Momentum conjugated to $q$ .
$q$	Angle defined by equation (15).
$q_0$	Initial value of $q$ .
$r$	Radius from the center of force to the center of mass.
$V$	Potential energy.
$T$	Kinetic energy.
$X, Y, Z$	Eccentricity perturbation functions.

$\theta$  True anomaly.

$\varphi$  Angle of libration.

$\Omega$  Constant  $\left[ = \sqrt{3} (1-e^2)^{\frac{1}{2}} \left( \frac{B-A}{C} \right)^{\frac{1}{2}} \right]$

/ Introduction -

This paper is devoted to studying the planar libration of a gravitationally oriented rigid satellite on an elliptic orbit. The equations describing the motion of the satellite are derived using the Lagrange method. The coupling of the orbital and librational motion does not permit the satellite's center of mass traveling on an elliptic orbit. By introduction of canonical variables the librational motion assumes a form particularly suited for the Von Zeipel technique. The mathematical treatment and the physical consideration in the present approach produce a solution which reveals the ultimate nature of satellite libration. The application of the canonical transformation method (1) to this type of problem is apparently new in the literature on satellite dynamics (2-14). Based on this analysis, a proposed damping method for satellite libration will be discussed in a later article.

## 2. Equations of Motion -

A satellite of mass  $m$  with principal axes  $\underline{i}$ ,  $\underline{j}$ , and  $\underline{k}$  at its center of mass is launched into an elliptic orbit. The longitudinal axis  $\underline{i}$  and the transverse axis  $\underline{j}$  lie in the orbital plane and the axis  $\underline{k}$  is normal to the orbital plane. The inertia tensor at the center of mass is

$$J = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \quad (1)$$

where  $A$ ,  $B$  and  $C$  are principal, central moments of inertia of the satellite about axes  $\underline{i}$ ,  $\underline{j}$  and  $\underline{k}$  respectively.

Let  $(r, \theta)$  be the polar coordinates of the satellite's center of mass and  $\phi$  which is in the orbital plane be the angle of inclination of the longitudinal axis  $\underline{i}$  to the line joining the center of force to the center of mass.

The kinetic energy of the satellite is

$$T = \frac{1}{2} m \left[ \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 \right] + \frac{1}{2} C \left[ \frac{d}{dt} (\theta + \phi) \right]^2 \quad (2)$$

By MacCullagh's formula, the potential energy is

$$V = -\frac{Km}{r} - \frac{K(A+B+C-3I)}{2r^3} + O\left(\frac{1}{r^4}\right) \quad (3)$$

in which K is the gravitational parameter of the center of force.

Thus for motion about the earth  $K = GMe$  where G is the constant of gravitation and  $Me$  is the mass of earth.  $I = \underline{n} \cdot \int \cdot \underline{n}$  and  $\underline{n}$  is a unit vector along  $\gamma$ . Accordingly,

$$I = A \cos^2 \phi + B \sin^2 \phi \quad (4)$$

To second order in  $\phi$  the Lagrangian of the motion is

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} m \left[ \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\theta}{dt} \right)^2 \right] + \frac{1}{2} C \left[ \frac{d}{dt} (\theta + \phi) \right]^2 \\ & + \frac{Km}{r} + \frac{K(B+C-2A)}{2r^3} - \frac{3K(B-A)}{2r^3} \phi^2 \end{aligned} \quad (5)$$

The three equations which govern the three degrees of freedom of the satellite motion are

$$\frac{d^2 r}{dt^2} - r \left( \frac{d\theta}{dt} \right)^2 + \frac{K}{r^2} + \frac{3K(B+C-3A)}{2mr^4} - \frac{9K(B-A)}{2mr^4} \phi^2 = 0 \quad (6)$$

$$\frac{d^2 \theta}{dt^2} + \frac{d^2 \phi}{dt^2} + 3 \frac{K}{r^3} \left( \frac{B-A}{C} \right) \phi = 0 \quad (7)$$

$$r^2 \frac{d^2 \theta}{dt^2} + 2r \left( \frac{dr}{dt} \right) \left( \frac{d\theta}{dt} \right) + \frac{C}{m} \left( \frac{d^2 \theta}{dt^2} + \frac{d^2 \phi}{dt^2} \right) = 0 \quad (8)$$

Equation (6) describes the radial motion of the center of mass.

Equation (7) could have been obtained directly by writing the rate of change of moment of momentum about the center of mass. Equation (8) expresses the conservation of moment of momentum about the center of force.

### 3. Librations about Elliptic Orbit -

The essentiality of the problem is contained in equations (6), (7) and (8). It consists in the coupling of the orbital and librational motion of the satellite. If one requires the center of mass to move on an elliptic orbit, i.e.

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \quad (9)$$

$$r^2 \left( \frac{d\theta}{dt} \right) = h \quad (10)$$

where  $h$  is an integral constant,  $a$  is the semi-major axis and  $e$  the eccentricity of the orbit, equation (8) becomes

$$\frac{C}{m} \left( \frac{d^2\theta}{dt^2} + \frac{d^2\phi}{dt^2} \right) \quad (11)$$

By substituting equation (11) in equation (7), the only solution to these equations is  $\phi = 0$ . This is expected physically because if  $\phi \neq 0$  the center of mass of the satellite is moving in a non-Newtonian force field which is not central. However, if  $\frac{C}{m}$  is much less than unity, it is acceptable to examine the librational and orbital motion separately.

Substituting equation (9) into equation (7) the result is

$$\frac{d^2\theta}{dt^2} + \frac{d^2\phi}{dt^2} + 3 \frac{K(1+e\cos\theta)^3}{a^3(1-e^2)^3} \left( \frac{B-A}{C} \right) \phi = 0 \quad (12)$$

Let the independent variable be transformed from the time to the polar coordinate, true anomaly  $\theta$ . One finds that

$$\frac{d^2\theta}{dt^2} + \frac{d^2\phi}{dt^2} = \frac{h^2}{r^4} \left[ -2 \frac{dr}{d\theta} \left( 1 + \frac{d\phi}{d\theta} \right) \frac{1}{r} + \frac{d^2\phi}{d\theta^2} \right] \quad (13)$$

Accordingly, equation (12) becomes

$$\frac{d^2\phi}{d\theta^2} - \frac{2e\sin\theta}{1+e\cos\theta} \frac{d\phi}{d\theta} + 3 \frac{1-e^2}{1+e\cos\theta} \left( \frac{B-A}{C} \right) \phi = \frac{4e\sin\theta}{1+e\cos\theta} \quad (14)$$

After introducing

$$\phi = \frac{9}{1+e\cos\theta} \quad (15)$$

equation (14) reduces to

$$\frac{dp}{d\theta} + \left[ \frac{3(1-e^2)(\frac{B-A}{C}) + e \cos \theta}{1 + e \cos \theta} \right] q = 4e \sin \theta \quad (16)$$

where  $p = \frac{dq}{d\theta}$  is the momentum conjugate to  $q$ .

Equation (16) can be expressed in the form

$$\frac{dp}{d\theta} + \Omega^2 q = (\Omega^2 - 1) \left[ \sum_{n=1}^{\infty} (-1)^{n+1} e^n \cos^n \theta \right] q + 4e \sin \theta \quad (17)$$

where  $\Omega^2 = 3(1-e^2)(\frac{B-A}{C}) > 0$ . For circular orbit, i.e.,  $e = 0$ , the librations will evidently be purely harmonic with constant frequency and the amplitude of libration will be constant over time depending on the initial conditions. With the existence of the eccentricity perturbation the solution of equation (17) is either almost periodic or diverging or decaying librational motion. A solution which can reveal the ultimate nature of the librational motion governed by equation (17) is desired.

#### 4. Canonical Transformation

The canonical equations corresponding to equation (17) are

$$\frac{dq}{d\theta} = \frac{\partial H}{\partial p} \quad (18)$$

$$\frac{dp}{d\theta} = -\frac{\partial H}{\partial q} - \mathcal{I} \quad (19)$$

where

$$H = \frac{1}{2} (p^2 + \Omega^2 q^2)$$

$$\mathcal{I} = -(\Omega^2 - 1)e(\cos\theta - e\cos^2\theta)q - 4e\sin\theta$$

H is the Hamiltonian of the unperturbed system and X is the perturbing function. In this expression, perturbation terms of order  $e^3$  and higher are omitted.

The solution for circular orbit subject to the transformed initial conditions  $q(0) = q_0$  and  $p(0) = 0$  can be expressed in periodic forms

$$q = q_0 \cos \Omega \theta \quad (20)$$

$$p = -q_0 \Omega \sin \Omega \theta \quad (21)$$

Define  $l = \Omega \theta$  and  $L = \frac{\Omega}{2} q_0^2$  so that equations (20) and (21) are transformed to

$$q = \sqrt{\frac{2L}{\Omega}} \cos l \quad (22)$$

$$p = -\sqrt{2L\Omega} \sin l \quad (23)$$

Since a transformation to new action and angle variables has been given in the form

$$H(p, q) = H'(L, l) = H'(L, -) = \omega L$$

the question is, under what circumstances will the transformation be canonical, i.e., under what circumstances the new equations will be

$$\frac{dl}{d\theta} = \frac{\partial H'}{\partial L} + Y \quad (24)$$

$$\frac{dL}{d\theta} = -\frac{\partial H'}{\partial l} - Z \quad (25)$$

This transformation of variables can be done in the following manner. Multiply equations (18) and (19) by  $\frac{\partial p}{\partial L}$ ,  $-\frac{\partial q}{\partial L}$  respectively and add. The sum is

$$\frac{dq}{dt} \frac{\partial p}{\partial L} - \frac{dp}{dt} \frac{\partial q}{\partial L} = \frac{\partial H}{\partial p} \frac{\partial p}{\partial L} + \frac{\partial H}{\partial q} \frac{\partial q}{\partial L} + \mathcal{F} \frac{\partial q}{\partial L} \quad (26)$$

If equations (18) and 19) are multiplied by  $\frac{\partial p}{\partial L}$ ,  $-\frac{\partial q}{\partial L}$  and similar operations are carried out the result is

$$\frac{dq}{dt} \frac{\partial p}{\partial l} - \frac{dp}{dt} \frac{\partial q}{\partial l} = \frac{\partial H}{\partial p} \frac{\partial p}{\partial l} + \frac{\partial H}{\partial q} \frac{\partial q}{\partial l} + \mathcal{F} \frac{\partial q}{\partial l} \quad (27)$$

Since

$$\frac{dq}{dt} = \frac{\partial q}{\partial l} \frac{dl}{dt} + \frac{\partial q}{\partial L} \frac{dL}{dt}$$

$$\frac{dp}{dt} = \frac{\partial p}{\partial l} \frac{dl}{dt} + \frac{\partial p}{\partial L} \frac{dL}{dt}$$

and

$$\frac{\partial H}{\partial p} \frac{\partial p}{\partial l} + \frac{\partial H}{\partial q} \frac{\partial q}{\partial l} = \frac{\partial H}{\partial l}$$

equations (26) and (27) become

$$\left[ \frac{\partial q}{\partial l} \frac{\partial p}{\partial L} - \frac{\partial p}{\partial l} \frac{\partial q}{\partial L} \right] \frac{dl}{d\theta} = \frac{\partial H'}{\partial L} + \sum \frac{\partial q}{\partial L} \quad (28)$$

$$\left[ \frac{\partial q}{\partial l} \frac{\partial p}{\partial L} - \frac{\partial p}{\partial l} \frac{\partial q}{\partial L} \right] \frac{dL}{d\theta} = -\frac{\partial H'}{\partial l} - \sum \frac{\partial q}{\partial l} \quad (29)$$

The Lagrangian bracket  $[l, L]$  in equations (28) and (29) equals unity because the Hamiltonian is unchanged. Therefore, comparing equations (28) and (29) with equations (24) and (25), one obtains

$$Y = \sum \frac{\partial q}{\partial L} \quad (30)$$

$$Z = \sum \frac{\partial q}{\partial l} \quad (31)$$

Differentiating equations (22) and (23) the results are

$$\frac{\partial \eta}{\partial l} = \frac{P}{\Omega} \quad \text{and} \quad \frac{\partial \eta}{\partial L} = \frac{\cos^2 l}{\Omega \eta}$$

By substituting these results equations (30) and (31) become

$$Y = - \left[ \frac{(\Omega^2 - 1)}{\Omega} e (\cos \theta - e \cos^2 \theta) \cos^2 l + \frac{4}{\Omega \eta_0} e \sin \theta \cos l \right] \quad (32)$$

$$Z = \left[ (\Omega^2 - 1) e (\cos \theta - e \cos^2 \theta) \eta_0^2 \cos l \sin l + 4 e \sin \theta \sin l \right] \quad (33)$$

## 5. General Solution

Thus the solution for the librational motion about an elliptic orbit is described by

$$\phi = \frac{\sqrt{2L}}{1+e\cos\theta} \cos l$$

where  $L$  and  $l$  are given by

$$\frac{dl}{d\theta} = -e \left[ (\Omega^2 - 1)(1 - e\cos\theta) \frac{g^2}{f_0} \cos\theta \cos l \sin l + \frac{4g}{f_0} \sin\theta \sin l \right] \quad (34)$$

$$\frac{dl}{d\theta} = \Omega - e \left[ \frac{(\Omega^2 - 1)}{\Omega} (1 - e\cos\theta) \cos\theta \cos^2 l + \frac{4}{\Omega f_0} \sin\theta \cos l \right] \quad (35)$$

The physical interpretation of these results is as follows. The amplitude and frequency of satellite librations on an elliptic orbit can be determined from the librations in a circular orbit by considering that

the amplitude and frequency are varying due to the eccentricity perturbation. The variation in amplitude is inherent in equation (34). The variation in frequency is given by equation (35). It should be noted that the last term in equation (34) represents secular eccentricity effect if  $\theta$  is equal to  $l$ . Accordingly, the amplitude will build up without bound in case  $\Omega = 1$ . For  $\Omega \neq 0$ , the results show that the amplitude to have no secular terms. It does not increase or decrease without bound but is represented by a constant plus trigonometric terms having frequencies which are fractions of the fundamental frequency of the system.

By excluding the periodic perturbations equations (34) and (35) reduce to

$$\left. \frac{dL}{d\theta} \right|_{SEC} = 0 \quad \text{for } \Omega \neq 1$$

or

$$= -2\frac{e}{\Omega} \quad \text{for } \Omega = 1.$$

and

$$\left. \frac{dl}{d\theta} \right|_{SEC} = \Omega + \frac{\Omega^2 - 1}{4\Omega} e^2$$

Taking these values in order to obtain a first approximation to the solution one obtains

$$\lambda = \left[ \Omega + \frac{\Omega^2 - 1}{4\Omega} e^2 \right] \theta$$

$$g^* = g_0 \text{ (constant)} \quad \text{for } \Omega \neq 1$$

$$\text{or} \quad = g_0 - 2e\theta \quad \text{for } \Omega = 1.$$

Therefore the first approximation of the librations is

$$\phi = \frac{g^*}{1 + e \cos \theta} \cos \left[ \Omega + \frac{\Omega^2 - 1}{4\Omega} e^2 \right] \theta \quad (36)$$

For  $\Omega = 1$ , equation (17) is reduced to a form with constant coefficients and the integral solution increases without bound. According to the physical sense of the moments of inertia and in view of the instability conditions this is referred to as the resonance case.

Equation (36) describes the librational motion of artificial satellites. The ultimate nature of the libration is concisely revealed. The eccentricity effect upon the amplitude and the eccentricity frequency shift are exhibited. It can be shown that the solution represented by equation (36) meets the limiting condition imposed. For circular orbit, i.e.,  $e = 0$ , equation (36) reduces to a solution which abounds in the literature of satellite dynamics. Based on this analysis, a proposed method which damps the librations and prevents librations from becoming arbitrarily large will be discussed in a later article.

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